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THE FUNDAMENTAL GROUP OF A MODULI SPACE OF BUNDLES ON $\mathbb{P}^3$

P. E. NEWSTEAD

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§1. INTRODUCTION

HARTSHORNE[1, Theorem 4.5; 2, Theorem 9.7] has given an explicit description of the moduli space $M(0,2)$ of stable bundles of rank 2 on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 2$. This description is valid over any algebraically closed field k, but in this paper we shall restrict attention to the case $k = \mathbb{C}$ and prove

THEOREM 1. *$M(0,2)$ is connected and $\pi_1(M(0,2)) \cong \mathbb{Z}/6$.*

It should be emphasized that our arguments depend entirely on Hartshorne's description, and therefore give no information about $M(0,n)$ for $n \geq 3$. (It is known that $M(0,n)$ is empty if $n \leq 0$ [2, Corollary 8.4], and that $M(0,1)$ is isomorphic to the complement of a non-singular quadric hypersurface in $\mathbb{P}^5$ [2, Example 8.4.1]; hence $M(0,1)$ is connected and $\pi_1(M(0,1)) \cong \mathbb{Z}/2$).

The subspace $I(2)$ of $M(0,2)$ consisting of those points of $M(0,2)$ which correspond to instantons has also been described explicitly [1, §4; 3, §4], and it is known that $I(2)$ is connected and $\pi_1(I(2)) \cong \mathbb{Z}/2$ [3, §4.2.2 and §4.4]. (This has also been proved independently by Rawnsley.) We shall prove

THEOREM 2. *The inclusion of $I(2)$ in $M(0,2)$ maps the generator of $\pi_1(I(2))$ to the unique element of order 2 in $\pi_1(M(0,2))$.*

It has been conjectured that $I(n)$ is connected and $\pi_1(I(n)) \cong \mathbb{Z}/2$ for all $n \geq 2$. On the other hand it is known that $M(0,n)$ is not connected for $n \geq 3$; however it is possible that a suitably stated analogue of Theorem 2 remains true. For the corresponding problem on $\mathbb{P}^2$, the moduli space is always connected, and le Potier[4, Proposition 6] has shown that it is simply-connected for $n \geq 3$, but has fundamental group $\mathbb{Z}/3$ for $n = 2$. It would be interesting to relate this last result to our Theorem. 1.

We work throughout over the complex numbers; all homology and cohomology groups have integral coefficients. We have systematically suppressed mention of the base points with respect to which the fundamental groups of various spaces are taken; all the spaces involved are connected.

The idea for this paper arose in discussions at a conference on algebraic vector bundles on projective spaces held in Oxford in May 1978. I am most grateful to Michael Atiyah and the other participants in this conference.

§2. OUTLINE OF PROOF OF THEOREM 1

From now on, we shall for convenience write M in place of $M(0,2)$. It follows from [2, Theorem 9.7] that points of M correspond precisely to pairs (ρ, p), where ρ is a regulus (i.e. one system of generators of a non-singular quadric in $\mathbb{P}^3$) and p is a linear system of degree 3 and dimension 1 on ρ without base points. In this way M

can be represented as a fibration

$$\begin{array}{c} U \to M \\ \downarrow \\ B, \end{array} \tag{1}$$

in which the base space B is a variety whose points correspond to reguli, and the fibre U is a variety whose points correspond to pencils of binary cubics without base points. The fibration (1) is locally trivial with respect to the usual topology, but not necessarily algebraically; we shall give a more precise description in §4.

We prove Theorem 1 by showing

(a) M is connected and $\pi_1(M)$ has order at most 6;

(b) $H_1(M) \cong Z/6$.

For (a) we prove

(a)′ B is connected and $\pi_1(B) \cong Z/2$;

(a)″ U is connected and $\pi_1(U) \cong Z/3$.

The assertion (a) now follows from the exact homotopy sequence of (1).

For (b) we use the homology spectral sequence of (1). First we need to prove

(b)′ $\pi_1(B)$ acts trivially on the homology of U.

Given this, we then obtain from the spectral sequence an exact sequence

$$H_2(B) \to H_1(U) \to H_1(M) \to H_1(B) \to 0.$$

By (a)′ and (a)″ we have $H_1(U) \cong Z/3$ and $H_1(B) \cong Z/2$. It therefore remains to show that the homomorphism $H_2(B) \to H_1(U)$ is zero, and for this it is sufficient to prove

(b)″ $H_2(B) \cong Z/2$.

It remains therefore to prove the assertions (a)′, (a)″, (b)′ and (b)″.

§3. THE BASE SPACE: PROOF OF (a)′ AND (b)″

Since every non-singular quadric in $\mathbb{P}^3$ contains precisely two reguli, the base space B of (1) can be represented as an unramified double covering of the space of such quadrics. This space can be identified with $\mathbb{P}^9 - \Delta$, where Δ is the quartic hypersurface given by equating the determinant of a general 4×4 symmetric matrix to zero. Moreover, it is well known (and easy to verify) that the singular set Σ of Δ is precisely the set of points which correspond to quadrics of rank ≤ 2. It follows that Σ has codimension 2 in Δ, so that the generic plane section of Δ is a non-singular quartic curve; hence $\pi_1(\mathbb{P}^9 - \Delta) \cong Z/4$ by Zariski's Theorem [5, 9 and 10]. To complete the proof of (a)′, it is therefore sufficient to prove that B is connected; in fact this is classical and quite easy to prove, but we shall in any case give another proof in the course of the next part of the argument.

For computations of homology, a slightly different description of B is more useful. For this, let $G = G(1,3)$ denote the Grassmannian of lines in $\mathbb{P}^3$. We can embed G in the usual way as a quadric hypersurface in $\mathbb{P}^5$; a regulus then becomes a non-singular conic on G whose plane does not lie entirely on G. It follows that

$$B \cong G(2,5) - H,$$

where $G(2,5)$ is the Grassmannian of planes in $\mathbf{P}^5$, and H is the subvariety whose points correspond to planes which touch G.

The structure of the variety $G(2,5)$ is well known; in particular it is irreducible and non-singular and has dimension 9. It follows that B is connected (as asserted earlier) and, by Lefschetz duality,

$$H_i(B) \cong H^{18-i}(G(2,5), H).$$

We wish to use this to compute $H_2(B)$; for this we consider the exact sequence

$$H^{15}(G(2,5)) \to H^{15}(H) \to H^{16}(G(2,5), H) \to H^{16}(G(2,5)) \to H^{16}(H).$$

In this sequence we know that

$$H^{15}(G(2,5)) = 0, \quad H^{16}(G(2,5)) \cong \mathbf{Z}.$$

Moreover H can be defined by a single equation in the Plücker coordinates on $G(2,5)$, and therefore has dimension 8; so the homomorphism

$$H^{16}(G(2,5)) \to H^{16}(H)$$

is injective. It follows that

$$H_2(B) \cong H^{16}(G(2,5), H) \cong H^{15}(H). \tag{2}$$

To complete the computation of $H_2(B)$, we look more closely at the structure of H. First let H' denote the subvariety of H whose points correspond to planes in $\mathbf{P}^5$ which either lie in G or meet G in a repeated line. Next we write

$$P = \{(p, \pi) \in G \times H : \pi \text{ touches } G \text{ at } P\}.$$

One can easily check that the projection $P \to G$ makes P into an algebraic bundle over G whose fibre is also G. (One can regard G as the Grassmannian of planes through the origin in $\mathbf{C}^4$; we can then identify P with the total space of the bundle with fibre G associated to the tangent $\mathbf{C}^4$-bundle of G.) In particular P is irreducible and non-singular, and has dimension 8.

We consider now the projection $f: P \to H$ and write $P' = f^{-1}(H')$. It follows from the definition of H' that f maps $P - P'$ bijectively, and therefore homeomorphically, to $H - H'$; in other words,

$$f: P, P' \to H, H'$$

is a relative homeomorphism. Moreover the fibres of f over points of H' have positive dimension; hence

$$\dim H' < \dim P' \leq 7,$$

since P is an irreducible variety of dimension 8 and $P' \neq P$. So

$$H^{15}(H) \cong H^{15}(H, H') \cong H^{15}(P, P') \cong H_1(P - P'), \tag{3}$$

the last isomorphism being given by Lefschetz duality.

One can check that the restriction to $P - P'$ of the projection $P \to G$ makes $P - P'$ into a fibration over G with fibre $G - G'$, where G' is the subvariety of G whose points correspond to lines in $\mathbb{P}^3$ which touch a given non-singular quadric; this fibration is locally trivial in the usual topology, though possibly not algebraically. Since G is simply-connected, the spectral sequence of this fibration gives rise to an exact sequence

$$H_2(P - P') \to H_2(G) \to H_1(G - G') \to H_1(P - P') \to 0. \tag{4}$$

It is well known (and easy to verify) that G' is embedded in $\mathbb{P}^5$ as an irreducible quartic threefold. Hence $H^6(G') \cong \mathbf{Z}$ and the image of the homomorphism

$$H^6(\mathbb{P}^5) \to H^6(G')$$

has index 4. On the other hand $H^6(G) \cong \mathbf{Z}$ and the image of the homomorphism

$$H^6(\mathbb{P}^5) \to H^6(G)$$

has index 2. It follows that the image of the homomorphism

$$H^6(G) \to H^6(G')$$

has index 2, and therefore that $H^7(G, G') \cong \mathbf{Z}/2$. Hence by Lefschetz duality

$$H_1(G - G') \cong \mathbf{Z}/2. \tag{5}$$

Finally we consider the homomorphism

$$H_2(P - P') \to H_2(G) \tag{6}$$

in (4). Let L be any line lying in G, and let L' be a line in $\mathbb{P}^5$ which does not meet the 3-space conjugate to L. For any point p of L, the tangent space to G at p meets L' in a unique point $\phi(p)$; let $\pi(p)$ be the plane which joins L to $\phi(p)$. The map

$$L \to P : p \mapsto (p, \pi(p))$$

then maps L into $P - P'$, and is therefore a section of $P - P'$ over L. Since the homomorphism

$$H_2(L) \to H_2(G)$$

is certainly surjective, so also is (6). Combining this fact with (2)–(5), we see that $H_2(B) \cong \mathbf{Z}/2$; this completes the proof of (b)″.

§4. THE FIBRATION: PROOF OF (b)′

We are now ready to describe the fibration (1) more precisely. We identify B with $G(2, 5) - H$ as in §3, and write

$$C = \{(p, \pi) \in G \times B : p \in \pi\}.$$

The fibres of the projection $C \to B$ are non-singular conics; one can check (by writing down equations) that C is topologically a $\mathbf{P}^1$-bundle over B. (Algebraically C is a "conic bundle", but is not locally trivial; topologically it is a $\mathbf{P}^1$-bundle which is not associated with any $\mathbf{C}^2$-bundle. These facts follow from [6, Proposition 8.1] and [7, Theorem].) There is a natural action of $PGL(2)$ on the space U of pencils of binary cubics without base points; this allows us to associate with any $\mathbf{P}^1$-bundle a corresponding bundle with fibre U. The space M can be identified with the bundle obtained from $\mathbf{C}$ in this way.

In order to prove (b)′, we shall construct a circle in B which carries the generator of $\pi_1(B)$, and prove that the restriction to this circle of the $\mathbf{P}^1$-bundle C is trivial; this implies that the same holds for the restriction of (1), and therefore that $\pi_1(B)$ operates trivially on the homology of U as required.

First we choose a non-singular point q of Δ, and take a line L in $\mathbf{P}^9$ through q and not lying in the tangent space to Δ at q. The line L meets Δ in a finite set of points; we choose a disc D in L, centre q, which does not meet Δ in any other point. The boundary of D then represents a generator of $\pi_1(\mathbf{P}^9-\Delta)$.

Recall that we are thinking of $\mathbf{P}^9$ as the space of quadrics in $\mathbf{P}^3$; thus a point of $\mathbf{P}^9$ corresponds to a quadratic form $\Sigma a_{ij}x_ix_j$. Let q be the point corresponding to

$$x_0x_1+x_2^2.$$

This is a non-singular point of Δ; in fact, the tangent space to Δ at q is given by $a_{33}=0$. We can therefore take for D the subset of $\mathbf{P}^9$ consisting of points of the form

$$x_0x_1+x_2^2-\lambda x_3^2,\quad |\lambda|\leq 1.$$

Since $\pi_1(B)$ has index 2 in $\pi_1(\mathbf{P}^9-\Delta)$, the generator of $\pi_1(B)$ maps to the element of $\pi_1(\mathbf{P}^9-\Delta)$ given by the circle

$$S^1\to\mathbf{P}^9-\Delta\colon \lambda\mapsto x_0x_1+x_2^2-\lambda^2x_3^2. \tag{7}$$

For each λ, the lines

$$\mu x_0+\nu(x_2-\lambda x_3)=\nu x_1-\mu(x_2+\lambda x_3)=0 \tag{8}$$

(μ, ν not both zero) form one of the two reguli on the quadric

$$x_0x_1+x_2^2-\lambda^2x_3^2=0;$$

this provides a lifting of the circle (7) to B. Let $p(\lambda,\mu,\nu)$ denote the point of G corresponding to the line (8); its Plücker coordinates are easily calculated and are given by

$$p_{01}=-2\lambda\mu\nu,\quad p_{02}=-\lambda\nu^2,\quad p_{03}=-\nu^2$$

$$p_{23}=\mu\nu,\quad p_{31}=-\mu^2,\quad p_{12}=-\lambda\mu^2.$$

The map

$$S^1\times\mathbf{P}^1\to G\colon(\lambda,(\mu,\nu))\to p(\lambda,\mu,\nu)$$

now provides the required trivialisation of the restriction of C to a circle representing a generator of $\pi_1(B)$.

§5. THE FIBRE: PROOF OF (a)″

We can identify the space of all binary cubics with $\mathbb{P}^3$ by associating the point (a_0, a_1, a_2, a_3) with the cubic

$$a_0t_0^3 + a_1t_0^2t_1 + a_2t_0t_1^2 + a_3t_1^3.$$

A pencil of binary cubics then becomes a line in $\mathbb{P}^3$, so that the space of all such pencils can be identified with G. To say that a pencil has a base point at (α, β) is equivalent to saying that the corresponding line lies entirely in the plane

$$\alpha^3x_0 + \alpha^2\beta x_1 + \alpha\beta^2x_2 + \beta^3x_3 = 0 \tag{9}$$

in $\mathbf{P}^3$. Thus the pencils with base point determine the subvariety X of G given by the lines which lie in the plane (9) for some $(\alpha, \beta) \in \mathbb{P}^1$. Clearly $U = G - X$; hence U is connected.

It is easy to check that X is an irreducible threefold. Moreover, a general quadric in $\mathbf{P}^3$ has just 6 of the planes (9) as tangent planes, and each such plane contains one generator of each system. It follows that the general regulus meets X in 6 points; thus X has degree 6 in $\mathbf{P}^5$. The argument used to prove (5) in §3 now shows that

$$H_1(U) \cong \mathbf{Z}/3.$$

It remains now to prove that $\pi_1(U)$ is abelian. For this, we consider the projection of$G$ from a point $p \in G$ to a hyperplane Π. Let T denote the tangent hyperplane to G at P, and let X', T' be the projections of X, T respectively. Since G is a quadric, the projection induces a homeomorphism

$$G - X - (G \cap T) \to \Pi - X' - T'.$$

Moreover, since G is non-singular, the natural homomorphism

$$\pi_1(G - X - (G \cap T)) \to \pi_1(G - X)$$

is surjective; it is therefore sufficient to show that $\pi_1(\Pi - X' - T')$ is abelian.

LEMMA. *With the above notation, there exists a point p of G such that the general plane π in Π meets $X' \cup T'$ in a curve Y whose only singularities are nodes and every component to which has degree* ≤ 4.

Before proving this lemma, we note that it is sufficient to give the required result. In fact $\pi - Y$ has abelian fundamental group by [8, Corollaries 1 and 2], and our result follows by Zariski's Theorem.

Proof of lemma. Our proof proceeds by the direct method of writing down equations for X and X'. For any line in $\mathbf{P}^3$, we can form a skew-symmetric matrix (p_{ij}) from its Plücker coordinates. This matrix has rank 2 and every non-zero row determines a point of the line; it follows that the equation of X may be obtained by eliminating α, β from the matrix equation

$$(p_{ij})\begin{pmatrix}\alpha^3\\ \alpha^2\beta\\ \alpha\beta^2\\ \beta^3\end{pmatrix} = 0.$$

Using also the identity

$$p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12}=0,$$

we see easily that X is given as a subvariety of G by the equation

$$p_{01}p_{31}^2+p_{23}p_{02}^2+(p_{12}+p_{03})(p_{03}^2-p_{01}p_{23})+2p_{02}p_{03}p_{31}=0.$$

Now take p to be the point of G given by

$$p_{12}=1, \quad p_{01}=p_{02}=p_{03}=p_{31}=p_{23}=0,$$

and Π to be the hyperplane $p_{12}=0$. Then T' is the hyperplane $p_{03}=0$ in Π, while X' is given by the equation

$$(p_{03}^2-p_{01}p_{23})(p_{01}p_{23}+p_{02}p_{31})=p_{03}(p_{01}p_{31}^2+p_{23}p_{02}^2+p_{03}(p_{03}^2-p_{01}p_{23})+2p_{02}p_{03}p_{31})$$

obtained by eliminating p_{12} from the equations of X and G.

Note that $X'\cap T'$ is given by

$$p_{01}p_{23}(p_{01}p_{23}+p_{02}p_{31})=0;$$

it follows that the general plane π in Π meets T' in a line which intersects X' in 4 distinct points. It is therefore sufficient to prove that the general π meets X' in a quartic curve whose only singularities are nodes. For this it is sufficient to prove that there exists a plane π_0 with this property, and this is certainly true for a suitable plane in T'. For example, one could take π_0 to be defined by

$$p_{03}=p_{01}+p_{23}+p_{02}+p_{31}=0.$$

This completes the proof of the lemma and hence of Theorem 1.

§6. PROOF OF THEOREM 2

Consider the composite map

$$I(2)\to M\to B\to \mathbf{P}^9-\Delta,$$

formed from the inclusion of $I(2)$ in M, the fibration (1), and the map $B\to\mathbf{P}^9-\Delta$ described in §3. Let R denote the image of $I(2)$ in $\mathbf{P}^9-\Delta$; according to [3, §4.4], the map $I(2)\to R$ is a fibration with contractible fibre. To prove Theorem 2 it is therefore sufficient to show that there exists a circle in R which represents the element of order 2 in $\pi_1(\mathbf{P}^9-\Delta)$.

Again by [3, §4.4], the space R is the subspace of $\mathbf{P}^9-\Delta$ whose points correspond to quadric surfaces which are invariant under the map

$$\mathbf{P}^3\to\mathbf{P}^3\colon (x_0,x_1,x_2,x_3)\mapsto(-\bar{x}_1,\bar{x}_0,-\bar{x}_3,\bar{x}_2).$$

A simple calculation shows that the circle

$$S^1\to\mathbf{P}^9-\Delta\colon \lambda\mapsto x_0^2+\ x_2^2+\lambda(x_1^2+x_3^2)$$

is contained in R. Moreover the map

$$S^1 \times [0,1] \to \mathbb{P}^9 - \Delta$$

$$(\lambda, r) \mapsto x_0^2 + x_2^2 + (\lambda - r^2\lambda)(x_1^2 + x_3^2) + r\lambda^2(x_1 - ix_3)^2 - r(x_1 + ix_3)^2$$

provides a homotopy from this circle to the circle

$$S^1 \to \mathbb{P}^9 - \Delta \colon \lambda \mapsto x_0^2 + x_2^2 + \lambda^2(x_1 - ix_3)^2 - (x_1 + ix_3)^2.$$

By an argument similar to the one used in §4 to construct a generator of $\pi_1(B)$, it follows that the given circle does indeed represent the element of order 2 in $\pi_1(\mathbb{P}^9 - \Delta)$.

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University of Liverpool
Liverpool
England